Numbness. **The Separation Theorem**.
Minimum Distance to a Convex Set.

Theorem: *Let x be a vector in a Hilbert Space H and let K be a closed convex subspace of H* (**as Interior**). *Then there is a unique vector* $k_0 \in K$ (**Selection**) *such that* $\|x - k_0\| \leq \|x - k\|$ *for all* $k \in K$. *Furthermore, a neccessary and sufficient condition that* k_0 *be a unique minimizing vector is that* $(x - k_0 \mid k - k_0) \leq 0$ *for all* $k \in K$. (**Lésions de Pression**)

Proof:

To prove the existence. Let $\{k_i\}$ be a sequence from K such that

$$\|x - k_i\| \to \delta = \inf_{k \in K} \|x - k\|$$

By the parallelogram Law

$$\|k_i - k_j\|^2 = 2\|k_i - x\|^2 + 2\|k_j - x\|^2 - 4\left\|x - \frac{k_1 + k_j}{2}\right\|^2$$

By convexity of K, $\frac{k_1+k_j}{2}$ is in K; hence

$$\left\|x - \frac{k_1 + k_j}{2}\right\| \geq \delta$$

and therefore $\|k_i - k_j\|^2 \leq 2\|k_i - x\|^2 + 2\|k_j - x\|^2 - 4\delta^2 \to 0$.

The sequence $\{k_i\}$ is Cauchy and hence converging to an element $k_0 \in K$. By continuity $\|x - k_0\| = \delta$.

To prove uniqueness, suppose $k_1 \in K$ with $\|x - k_1\| = \delta$. The sequence $k_n = \begin{bmatrix} k_0 \text{ for } n \text{ even} \\ k_1 \text{for } n \text{ odd} \end{bmatrix}$ has $\|x - k_n\| \to \delta$ so, by the above argument, $\{k_i\}$ is Cauchy and convergent. This can only happen if $k_1 = k_0$.

We show now that if k_0 is a unique minimizing vector, then $(x - k_0 \mid k - k_0) \leq 0$ for all $k \in K$. Suppose to the contrary that there is avector $k_1 \in K$ such that $(x - k_0 \mid k_1 - k_0) = \epsilon > 0$. Consider the vectors $k_\alpha = (1 - \alpha)k_0 + \alpha k_1$; $0 \leq \alpha \leq 1$. Since K is convex, each $k_\alpha \in K$. Also

$$\|x - k_\alpha\|^2 = \|(1 - \alpha)(x - k_0) + \alpha(x - k_1)\|^2$$

$$= (1 - \alpha)^2\|x - k_0\|^2 + 2\alpha(1 - \alpha)(x - k_0 \mid x - k_1) + \alpha^2\|x - k_1\|^2$$

The quantity $\|x - k_\alpha\|^2$ is a differentiable function of α with derivative at $\alpha = 0$ equal to

$$\frac{d}{d\alpha}\|x - k_\alpha\|^2 \mid_{\alpha=0} = -2\|x - k_0\|^2 + 2(x - k_0 \mid x - k_1)$$

$$= 2(x - k_0 \mid k_1 - k_0) = -2\epsilon < 0$$

Thus for some small positive α, $\|x - k_\alpha\| < \|x - k_0\|$ which contradict the minimizing property of k_0. Hence, no such k_1 can exist.

Conversley, suppose that $k_0 \in K$ is such that $(x - k_0 \mid k - k_0) \leq 0$ for all $k \in K$. Then for any $k \in K$, $k \neq k_0$, we have

$$\|x - k\|^2 = \|x - k_0 + k_0 - k\|^2 =$$

$$= \|x - k_0\|^2 + 2(x - k_0 \mid k_1 - k) + \|k_0 - k\|^2 > \|x - k_0\|^2$$

and therefore k_0 is a unique minimizing vector. QED.