Dual Spaces and Adjunct Operators.

Generalities on Functionals: $\exists V$, and $\phi : V \to \mathbb{R}$, $\forall a,b \in \mathbb{R}, u,v \in V$, $\phi(au+bv) = a\phi(u) + b\phi(v)$.

The Selective Linear Functional: $\pi_i : \mathbb{R}^n \to \mathbb{R}, \pi_i(a_i) = a_i$

The Vector Space on Polynomials over t: $J : V \to \mathbb{R}$, $J(p(t)) = \int_0^1 p(t)dt$ with $J(ap(t) + bp'(t)) = aJ(p(t)) + bJ(p'(t))$

We have Integration and Trace on Eigenvalues: $T : V \to \mathbb{R} \quad T(A) = a_{11} + a_{22} + \ldots + a_{nn}$, $(a_{ij}) = A$

About Spaces: If $\exists V, V'$, then $A : V \to V'$, $A = (a_{ij})$ is also a vector space $\|Hom(V,V')\| = nm$, $\|V\| = n$, $\|V'\| = m$.

Definitions: If $V = \mathbb{R}^n$, $\phi(a_1,..,a_n)$, $\phi : V \to V'$, $\phi : V \to V'$, $\phi(x_i) = (a_1,..,a_n)(x_i)$ a row and column. We call V^* Dual of V as $(a_1,..,a_n) \in V$ and $\phi \in V^*$. ϕ is called functional (and is a function $\phi(t)$).

Dual Basis: if $V = \mathbb{R}^n$, $\|V\| = n$, $\|V'\| = m$, $\|V^*\| = n$ as $V^* = V$ and there we have a **Dual Basis**.

If $\{v_i\} \mid_{i=1,\ldots,n}$ spans V, and $\delta_{ij} = \left\{ \begin{array}{c} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{array} \right\} = \phi_i(v_j)$ then $\{\phi_j\} \mid_{j=1,..,n}$ is a basis for V^*.

Example: if $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$ span $\mathbb{R}^2$, and expect functions $\{\phi_1, \phi_2\}$ span $\mathbb{R}^{2*}$ and we know that $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \phi_1(x,y) \\ \phi_2(x,y) \end{bmatrix} = \{x,y\}$ with

$\delta_{11} = \phi_1(v_1) = \delta_{22} = \phi_{22}(v_2) = 1$, $\delta_{12} = \phi_1(v_2) = \delta_{21} = \phi_2(v_1) = 0$,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \left\{ \begin{array}{c} 2a + b = \delta_{11} \\ 2c + d = \delta_{21} \end{array} \right\}, \quad \phi_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2a + b = 1 \\ 2c + d = 0 \end{bmatrix} \text{ where}$$

$a = -1, b = 3$. $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \left\{ \begin{array}{c} 3a + b = \delta_{12} \\ 3c + d = \delta_{22} \end{array} \right\}$,

$\phi_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2c + d = 0 \\ 3c + d = 1 \end{bmatrix}$ where $c = 1, d = -2$.

$$\phi_1 \begin{bmatrix} x \\ y \end{bmatrix} . = -x + 3y, \quad \phi_2 \begin{bmatrix} x \\ y \end{bmatrix} . = x - 2y, \text{ and } \{\phi_i\} \text{ span } \mathbb{R}^{2*}.$$

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \leftrightarrow A^{-1} = \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \text{ or differently } Ax \leftrightarrow \phi = A^{-1}x$$

A Résumé: If $\{v_i\}$ span V, $\{\phi_i\}$ span V^*, $u \in V$, $u = \phi_1(u)v_1 + \phi_2(u)v_2 +..+\phi_n(u)v_n$. $\sigma \in V^*$, $\sigma = \sigma(v_1)\phi_1 + \sigma(v_2)\phi_2 +\ldots+\sigma(v_n)\phi_n$, $\sigma(v_i) \in \mathbb{R}$, ϕ_i is a function.

The Inner Product on $\mathbb{R}^n$: $\langle u,v\rangle = u^T v$.

The definition of the Adjunct Operator: $T : V \to V$, $\exists T^*$ adjunct as $\langle Tu,v\rangle = \langle u,T^* v\rangle$, $u,v \in V$.

Integration and Trace on Eigenvalues: T is square and n dimensional, then $\langle Tu,v\rangle = \langle u,T^T v\rangle$, and $T^T = T$.

If $V = \mathbb{R}^n$, $\phi = (a_1,..,a_n)$, $\phi : V \to V'$, $\phi(x_i) = (a_1,..a_n)(x_i)$, we call V^* dual of V as $(a_1,a_2,..a_n) \in V$ and $\phi \in V^*$. ϕ is also called functional (and is a function $\phi(t)$).

If $\{v_i\} \mid_{i=1,..,n}$ spans V, and $\delta_{ij} = \left\{ \begin{array}{c} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{array} \right\} = \phi_i(v_j)$ then $\{\phi_j\} \mid_{j=1,..,n}$ is a basis for V^*.

Inner product space V, if $u \in V$, $\exists \widehat{u} : \mathbb{R}^n \to \mathbb{R}$, by $\widehat{u}(v) = \langle v,u\rangle$. We call $\widehat{u}$ a linear functional on V, $\widehat{u} \in V^*$.

Example of inner product space V and $\widehat{u}$ a linear functional on V.

$$\begin{bmatrix} 3 & 4 & -5 \\ 2 & -6 & 7 \\ 5 & -9 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = F\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{array}{c} F_1. = \begin{bmatrix} 3 & 4 & -5 \end{bmatrix} \\ F_2. = \begin{bmatrix} 2 & -6 & 7 \end{bmatrix} \\ F_3. = \begin{bmatrix} 5 & -9 & 1 \end{bmatrix} \end{array} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \widehat{u}_1 \\ \widehat{u}_2 \\ \widehat{u}_3 \end{bmatrix}$$

$$F^T = \begin{bmatrix} 3 & 4 & 5 \\ 4 & -6 & 7 \\ -5 & +7 & 1 \end{bmatrix}\begin{bmatrix} \widehat{u}_1 \\ \widehat{u}_2 \\ \widehat{u}_3 \end{bmatrix} = \begin{bmatrix} 3u_1 + 4u_2 + 5u_3 \\ 4u_1 - 6u_2 + 7u_3 \\ 7u_2 - 5u_1 + u_3 \end{bmatrix}$$

$\langle Ax,y\rangle = \langle x,A^T y\rangle$

$\langle u,y\rangle = \langle A^{-1}u,A^T y\rangle = \langle x,A^T y\rangle$